# Graded Version of Some Basic Theorems on Local Cohomology to a Pair of Ideals

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**Abstract**

In this paper, we prove some well-known results on local cohomology with respect to a pair of ideals in graded version, such as, Independence Theorem, Lichtenbaum-Harshorne Vanishing Theorem, Basic Finiteness and Vanishing Theorem, among others. Besides, we present a generalized version of Melkersson Theorem about Artinianess of modules and a result concerning Artinianess of local cohomology modules.

## 1 Introduction

Local cohomology with respect to a pair of ideals was firstly defined in [TYY], where the authors generalized the usual notion of local cohomology module and studied its various properties such as the relation between the usual local cohomology module, $H^i_I(M)$, and the one defined to a pair of ideals, $H^i_{I,J}(M)$, vanishing and nonvanishing theorems, the Generalized Version of Lichtenbaum-Hartshorne Theorem, among others.

Having the above results as motivation, the aim of this paper is to present in graded version some basic theorems on cohomology with respect to a pair of ideals, such as, Independence Theorem, Lichtenbaum-Harshorne Vanishing Theorem, Basic Finiteness and Vanishing Theorems, and assertions concerning Artinianess and depth with respect to a pair of ideals.

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The organization of this paper is as follows.

Let $R$ be a graded ring and let $I \subseteq R$ be a graded ideal, $J$ an arbitrary ideal and $M$ a graded $R$-module. Let $R_+$ denote the irrelevant ideal of $R$, that is, the ideal generalized by elements of positive degree.

In section 2, we view the local cohomology ${}^*H^i_{I,J}(M)$ as a graded module and express it in terms of usual local cohomology modules. To be more precise, we denote by ${}^*\widetilde{W}(I,J)$ the set of homogeneous ideals $\mathfrak{c}$ of $R$ such that $I^n \subseteq \mathfrak{c} + J$ for some integer $n$, and then, show that

$${}^*H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}}(M).$$

We also present the graded version of the Independence Theorem and Lichtenbaum-Harshorne Vanishing theorem for a pair of ideals.

In section 3, we suppose $R$ is a positively graded Noetherian ring which is standard, that is, $R = R_0[R_1]$, where $R_0$ is local ring. In [RS, Theorem 2.1], the authors show that if $n = \sup\{i \ : H^i_{R_+}(M) \neq 0\}$ then the $R$-module $H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)$ is Artinian. We generalize this result for the case of local cohomology with respect to a pair of homogeneous ideals, besides showing that $H^i_{R_+,J}(M)/(\mathfrak{m}_0R + J)H^i_{R_+,J}(M)$ is actually Artinian for all $i \geq 0$. Furthermore, we prove that

$$\dim M/(\mathfrak{m}_0R + J)M = \sup\{i \ : H^i_{R_+,J}(M) \neq 0\},$$

a generalization for [BH, Lemma 3.4]. We also present a new version, with respect to a pair of ideals, for Melkersson's Theorem about Artinianess.

In section 4, the module $M$ and the ring $R$ are assumed to be a Cohen-Macaulay, and then, it is obtained an expression to the number

$$\inf\{i \in \mathbb{N}_0 | H^i_{R_+,J}(M) \neq 0\}.$$

In section 5, $R_0$ is assumed to be a local ring with infinite residual field. It is well-known that, for all $i \geq 0$, $H^i_{R_+}(M)_n$ is finitely generated $R_0$-module for $n \in \mathbb{Z}$ and $H^i_{R_+}(M)_n = 0$, for $n$ sufficiently large. We give a positive answer for this in the case of cohomology modules with respect to a pair of ideals. If $J$ is generated by elements of zero degree then, for $i \geq 1$, $H^i_{R_+,J}(M)_n = 0$, for $n$ sufficiently large and $H^i_{R_+,J}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$. Finally, we prove a result about asymptotical stable (Theorem 5.6).

## 2 Graded versions for a pair of ideals

In this section, we introduce a grading to $H^i_{I,J}(M)$, making this a graded module. The result [TYY, Theorem 3.2], Independence theorem for a pair of ideals and Generalized Version of Lichtenbaum-Hartshorne Theorem are again presented now from the point of view of graded modules.

(A) Let $R$ be a graded ring and let $I \subseteq R$ be a graded ideal and $J$ an arbitrary ideal. If $M$ is a graded $R$-module then $\Gamma_{I,J}(M)$ is a graded submodule of $M$. In fact, pick $m = (m_k) \in \Gamma_{I,J}(M)$, so $I^n \subseteq (0 : m) + J$. It is easy to see that $I^n \subseteq (0 : m_k) + J$, for all $i$ and each $k$. Then define $\Gamma_{I,J}(M)_i = \{m \in M_i : mI^n \subseteq mJ$ for some positive integer $n \geq 1\}$.

(B) For a homomorphism $f : M \to N$, we have $f(\Gamma_{I,J}(M)) \subseteq \Gamma_{I,J}(N)$, so that there is a mapping $\Gamma_{I,J}(f) : \Gamma_{I,J}(M) \to \Gamma_{I,J}(N)$, which is the restriction of $f$ to $\Gamma_{I,J}(M)$. Thus $\Gamma_{I,J}$ is an additive functor on the category of all graded $R$-modules.

(C) Since the category of the graded modules has enough injectives, we can form the $i$-th right derived functor of $\Gamma_{I,J}$ (on the category of the graded modules), which will be denoted by ${}^*H^i_{I,J}$ ($i \geq 0$). For a graded $R$-module $M$, we shall refer to ${}^*H^i_{I,J}(M)$ to be the $i$-th graded local cohomology module of $M$ with respect to the pair of ideals $(I, J)$.

(D) One can derive, by functor properties, that given an exact sequence $0 \to M \to N \to P \to 0$ of graded $R$-modules, there is a long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & {}^*H^0_{I,J}(M) & \to & {}^*H^0_{I,J}(N) & \to & {}^*H^0_{I,J}(P) & \to & \\ & \to & {}^*H^1_{I,J}(M) & \to & {}^*H^1_{I,J}(N) & \to & {}^*H^1_{I,J}(P) & \to & \cdots, \end{array}$$

of graded modules with respect to a pair of ideals.

**Definition 2.1.** *We denote by* ${}^*\widetilde{W}(I, J)$ *the set of homogeneous ideals* $\mathfrak{c}$ *of* $R$ *such that* $I^n \subseteq \mathfrak{c} + J$ *for some integer* $n$*. We also define a partial order for this set:*

$$\mathfrak{c} \leq \mathfrak{d} \ \textit{if}\ \mathfrak{c} \supseteq \mathfrak{d}, \ \textit{for}\ \mathfrak{c}, \mathfrak{d} \in {}^*\widetilde{W}(I, J).$$

*If* $\mathfrak{a} \leq \mathfrak{b}$ *we obtain the inclusion map* $\Gamma_{\mathfrak{a}}(M) \hookrightarrow \Gamma_{\mathfrak{b}}(M)$*. The order relation on* ${}^*\widetilde{W}(I, J)$ *and the inclusion maps turn* $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)}$ *into a direct system of graded* $R$*-modules.*

**Proposition 2.2.** *Let* $R$ *be a graded ring,* $I$ *a graded ideal,* $J$ *an arbitrary ideal of* $R$ *and* $M$ *a graded* $R$*-module. Then there is a natural graded isomorphism*

$${}^*H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c} \in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}}(M)$$

*Proof.* Firstly we observe that $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M)$. In fact, if $x \in \bigcup_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M)$, we have $I^m \subseteq \mathfrak{c} + J$ and $xI^n = 0$ for positive integers $n, m$ and some $\mathfrak{c} \in^* \widetilde{W}(I,J)$. Since $I^{mn} \subseteq (\mathfrak{c}+J)^n \subseteq \mathfrak{c}^n + J$, we have $I^{mn}x \subseteq Jx$, that is, $x \in \Gamma_{I,J}(M)$. Now let $x \in \Gamma_{I,J}(M)$ homogeneous. Then $I^n \subseteq \mathfrak{c} + J$, where $\mathfrak{c} = \operatorname{ann}(x)$, so $x\mathfrak{c} = 0$ and $x \in \Gamma_{\mathfrak{c}}(M)$. If $x$ is not homogeneous, we write $x = x_1 + ... + x_r$, where $x_i$ is homogeneous. We have then $\mathfrak{a}_i = \operatorname{ann}(x_i)$ for each $i$. Hence $x\mathfrak{a}_1 \cdots \mathfrak{a}_r = 0$. Therefore $x \in \Gamma_{\mathfrak{c}}(M)$, where $\mathfrak{c} = \mathfrak{a}_1 \cdots \mathfrak{a}_r$.

Let $0 \to M \to N \to P \to 0$ be an exact sequence of $R$-modules. For each $\mathfrak{c} \in^* \widetilde{W}(I,J)$, we have a long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & {}^*H^0_{\mathfrak{c}}(M) & \to & {}^*H^0_{\mathfrak{c}}(N) & \to & {}^*H^0_{\mathfrak{c}}(P) & \to & \\ & \to & {}^*H^1_{\mathfrak{c}}(M) & \to & {}^*H^1_{\mathfrak{c}}(N) & \to & {}^*H^1_{\mathfrak{c}}(P) & \to & \cdots, \end{array}$$

which we can take the limit to obtain a long exact sequence

$$\begin{array}{ccccccc} 0 & \to & \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^0_{\mathfrak{c}}(M) & \to & \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^0_{\mathfrak{c}}(N) & \to & \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^0_{\mathfrak{c}}(P) \\ & \to & \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^1_{\mathfrak{c}}(M) & \to & \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^1_{\mathfrak{c}}(N) & \to & \cdots. \end{array}$$

Since ${}^*H^i_{\mathfrak{c}}(E) = 0$ for any ${}^*$injective $R$-module $E$ and any positive integer $i$, $\varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}}(E) = 0$. Therefore, one may conclude that

$$\{\varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}} \quad | \; i = 0, 1, 2, ...\}$$

is a system of right derived functors of ${}^*\Gamma_{I,J}$ (see [BS, Theorem 12.3.1]), and that there is the desired graded isomorphism. $\square$

**Remark 2.3.** *Consider the above setup. Since $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M)$, one may conclude similarly to the above proof that*

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} H^i_{\mathfrak{c}}(M).$$

The following result is obtained from the previous proposition once ${}^*H^i_{\mathfrak{c}}(E) = 0$ for any graded ideal $\mathfrak{c}$ and an ${}^*$injective graded $R$-module $E$.

**Proposition 2.4.** *Let $R$ be a graded ring, $I$ a graded ideal and $J$ an arbitrary ideal. Let $E$ be an ${}^*$injective graded $R$-module. Then $H^i_{I,J}(E) = 0$.*

By Proposition 2.4, we have the following

**Proposition 2.5.** *Let $R$ be a graded ring, $I$ a graded ideal and $J$ an arbitrary ideal. Let $M$ be an $R$-graded module. There is an isomorphism*

$$^*H^i_{I,J}(M) \cong H^i_{I,J}(M),$$

*for all $i$ as underlying $R$-modules.*

**Theorem 2.6.** (Graded independence theorem for a pair of ideals)

*Assume $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring, $I \subset R$ a graded ideal and $J \subset R$ an arbitrary ideal. Let $R' = \bigoplus_{n\in\mathbb{Z}} R'_n$ be another Noetherian graded ring, $f : R \to R'$ a graded homomorphism of rings such that $f(I) = JR'$ and $M'$ an $R'$-module.*

1. *For each $i \in \mathbb{N}_0$, both the cohomology modules*

$$H^i_{IR',JR'}(M') \quad \text{and} \quad H^i_{I,J}(M')$$

*are graded $R$-modules;*

2. *For each $i \in \mathbb{N}_0$, there exists a graded isomorphism*

$$H^i_{IR',JR'}(M') \cong H^i_{I,J}(M')$$

*of graded $R$-modules.*

*Proof.* The first item follows because $f$ is homogeneous.

The assumption $f(I) = JR'$ gives $\Gamma^i_{IR',JR'}(M') = \Gamma^i_{I,J}(M')$. By [TYY, Theorem 2.7], for each graded $R$-module $M'$, one has an isomorphism

$$H^i_{IR',JR'}(M') \cong H^i_{I,J}(M').$$

By using the grading on $H^i_{I,J}(M')$ obtained from first item, we can turn this isomorphism into a graded isomorphism. On the other hand, given an $R$-graded homomorphism $f : M' \to N'$, we have a natural commutative diagram

$$\begin{array}{ccc} H^i_{I,J}(M') & \longrightarrow & H^i_{I,J}(N') \\ \downarrow & & \downarrow \\ H^i_{IR',JR'}(M') & \longrightarrow & H^i_{IR',JR'}(N'). \end{array}$$

Hence, as $H^i_{IR',JR'}(E)$ equals zero (by Proposition 2.4), these (new) gradings coincide with the ones from item *1*. So, one can conclude the second item. $\square$

**Lemma 2.7.** *Let $R$ be a graded local ring of unique graded maximal ideal $\mathfrak{m}$. Let $d$ be the dimension of $R$, $I$ a graded ideal and $J$ an arbitrary ideal of $R$. Then $H^d_{I,J}(R) \cong H^d_{I\widehat{R},J\widehat{R}}(\widehat{R})$.*

*Proof.* By [CW, Theorem 2.1] (the same proof is true for the non-local case), $H^d_{I,J}(R)$ is Artinian. So, by [TYY, Lemma 4.8], we have

$$H^d_{I,J}(R) \cong H^d_{I,J}(R) \otimes_R \widehat{R} \cong H^d_{I,J}(\widehat{R}).$$

On the other hand, by [TYY, Theorem 2.7], one has $H^d_{I,J}(\widehat{R}) \cong H^d_{I\widehat{R},J\widehat{R}}(\widehat{R})$. The lemma is then concluded. □

**Theorem 2.8.** (Graded Lichtenbaum-Hartshorne Vanishing Theorem to a pair of ideals)

*Let $R = \oplus_{n\geq 0} R_n$ be a positively graded ring and an integral domain of dimension $d$. Assume $R_0$ is a complete local ring. Let $I$ be a proper graded ideal and $J$ an arbitrary ideal of $R$. Then $H^d_{I,J}(R) = 0$.*

*Proof.* If $J = 0$, it reduces to the usual case, already proved. Suppose then $J$ is a nonzero ideal. Let $\mathfrak{m}$ be the unique graded maximal ideal of $R$. We know the completion $\widehat{R_\mathfrak{m}}$ is isomorphic to the $\mathfrak{m}$-adic completion $\widehat{R}$ of $R$. Further, since $R_0$ is complete, it is known that $\widehat{R}$ is a domain. By the generalized Lichtenbaum-Harshorne Vanishing theorem ([TYY, Theorem 4.9]) one can deduce $H^d_{IR_\mathfrak{m},JR_\mathfrak{m}}(R_\mathfrak{m}) = 0$. On the other hand, by Lemma 2.7, one obtains

$$H^d_{IR_\mathfrak{m},JR_\mathfrak{m}}(R_\mathfrak{m}) \cong H^d_{I,J}(R).$$

Therefore $H^d_{I,J}(R) = 0$. □

**Proposition 2.9.** ([TYY, Corollary 4.2]) *Let $R$ be a graded local ring of unique graded maximal ideal $\mathfrak{m}$ and $M$ be a finite graded module over $R$. Let $I, J$ be graded ideals of $R$. If $H^i_{I,J}(M) = 0$ for all integers $i > 0$, then $M$ is an $(I, J)$-torsion $R$-module.*

*Proof.* Set $N = M/\Gamma_{I,J}(M)$. We need to prove that $N = 0$. Suppose $N \neq 0$. By [TYY, Corollary 1.13], we have $\Gamma_{I,J}(N) = 0$ and $H_{I,J}(N) = H_{I,J}(M) = 0$ for all $i > 0$. Since by hypothesis $I, J$ are graded ideals, we have $\mathfrak{m} \in W(I, J)$, so that

$$\inf\{\operatorname{depth} N_\mathfrak{p} \mid \mathfrak{p} \in W(I, J)\} \leq \operatorname{depth} N_\mathfrak{m} = \operatorname{depth} N < \infty.$$

By using [TYY, Theorem 4.1], we obtain $H_{I,J}(N) \neq 0$ for some integer $i \leq \operatorname{depth} N$, which is a contradiction. □

# 3 Top local cohomology

In this section, we give a version for Melkersson's Theorem concerning Artinianess of module with respect to a pair of ideals (Proposition 3.2). Also it is obtained a result about Artinianess of local cohomology with respect to a pair of ideal (Theorem 3.3). To conclude the section, we find the top of the local cohomology with respect to a pair of ideals (Theorem 3.5).

Throughout this section, let $R = \oplus_{d\geq 0} R_d$ denote a positively graded commutative Noetherian ring, which is standard, that is, $R = R_0[R_1]$. Assume $R_0$ is a local ring of maximal ideal $\mathfrak{m}_0$. Set $R_+ = \oplus_{i>0} R_i$, the irrelevant ideal of $R$. Let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finite graded $R$-module. Let $M[a]$ denote the graded module $a$-shift of $M$, defined by $M[a]_i = M_{i+a}$.

**Remark 3.1.** *It can be derived from Proposition* 2.2 *that* $H^i_{I,J}(M[a]) \cong H^i_{I,J}(M)[a]$ *as homogeneous modules.*

**Proposition 3.2.** ([M, Theorem 1.3]) *Let* $M$ *be an* $(I, J)$*-torsion* $R$*-module for which* $(JM :_M I)$ *is an Artinian module. Then* $M$ *is an Artinian module.*

*Proof.* If $M$ is an $(I, J)$-torsion $R$-module, then $M/JM$ is an $I$-torsion module by [TYY, Corollary 1.9]. Since by assumption $(JM :_M I) = (0 :_{M/JM} I)$ is an Artinian module we can use a result due to Melkersson (see [M, Theorem 1.3]) to get $M/JM$ is an Artinian module. Moreover, $JM \subseteq (JM :_M I)$ is also an Artinian module. By an exact sequence one can conclude $M$ is an Artinian module. ☐

Next theorem is a generalization of Theorem 2.1 in [RS].

**Theorem 3.3.** *Let* $J$ *be a graded ideal of* $R$*. Then the* $R$*-module*

$$H^i_{R_+,J}(M)/(\mathfrak{m}_0 R + J)H^i_{R_+,J}(M)$$

*is Artinian for all* $i \geq 0$.

*Proof.* Since $M$ is a finite $R$-module one may verify $R^k_+\Gamma_{R_+,\mathfrak{b}}(M) \subset J\Gamma_{R_+,J}(M)$ for some integer $k$. It implies $(\mathfrak{m}_0 + R_+)^k\Gamma_{R_+,J}(M) \subset \mathfrak{m}_0\Gamma_{R_+,J}(M) + JH^n_{R_+,J}(M)$, so that one may conclude

$$\Gamma_{R_+,J}(M)/(\mathfrak{m}_0 R + J)\Gamma_{R_+,J}(M)$$

is Artinian.

Now assume, by induction, that $i > 0$ and we have shown

$$H^{i-1}_{R_+,J}(M')/(\mathfrak{m}_0R + J)H^{i-1}_{R_+,J}(M')$$

is an Artinian module for any finite graded $R$-module $M'$. In view of [TYY, Corollary 1.13], we can assume $M$ is an $(R_+, J)$-torsion free $R$-module. Since $\Gamma_{R_+}(M) \subseteq \Gamma_{R_+,J}(M)$, there exists a homogeneous element $x \in R_+$ which is $M$-regular. Say, $\deg(x) = a$. The exact sequence

$$0 \to M \xrightarrow{x} M[-a] \to (M/xM)[-a] \to 0$$

induces an exact sequence

$$\begin{aligned} &\cdots \to H^{i-1}_{R_+,J}(M/xM) \\ \to H^i_{R_+,J}(M) \xrightarrow{x} H^i_{R_+,J}(M)[-a] \to &H^i_{R_+,J}(M/xM)[-a] \to \cdots \end{aligned}$$

Hence, we have the exact sequence

$$\frac{H^{i-1}_{R_+,J}(M/xM)}{(\mathfrak{m}_0R + J)H^{i-1}_{R_+,J}(M/xM)} \to \frac{H^i_{R_+,J}(M)}{(\mathfrak{m}_0R + J)H^i_{R_+,J}(M)} \xrightarrow{x}$$

$$\to \frac{xH^i_{R_+,J}(M)[-a]}{x(\mathfrak{m}_0R + J)H^i_{R_+,J}(M))[-a]} \to 0.$$

By induction hypothesis, we have $\frac{H^{i-1}_{R_+,J}(M/xM)}{(\mathfrak{m}_0R+J)H^{i-1}_{R_+,J}(M/xM)}$ is Artinian. It shows the kernel of the multiplication by $x$ on $\frac{H^i_{R_+,J}(M)}{(\mathfrak{m}_0R+J)H^i_{R_+,J}(M)}$ is an Artinian $R$-module. By using [TYY, Corollary 1.13], one can deduce $\frac{H^i_{R_+,J}(M)}{(\mathfrak{m}_0R+J)H^i_{R_+,J}(M)}$ is an $((x), J)$-torsion $R$-module. To conclude the proof, we apply Proposition 3.2 in order to obtain $\frac{H^i_{R_+,J}(M)}{(\mathfrak{m}_0R+J)H^i_{R_+,J}(M)}$ is Artinian. This completes the inductive step.

$\square$

**Theorem 3.4.** *Let $J$ be a graded ideal of $R$. If $\dim M/(\mathfrak{m}_0R + J)M = d$, then*

$$H^i_{R_+,J}(M) = 0,$$

*for all $i > d$.*

*Proof.* We argue by induction. Let $n = \dim_R M$. Suppose $n = -1$, so there is nothing to be proved, as $M = 0$. Assume then $n \geq 0$ and that the result is established for $R$-modules of dimension smaller than $n$. By [TYY, Corollary 2.5], $H^i_{R_+,J}(\Gamma_J(M)) \cong H^i_{R_+}(\Gamma_J(M))$ for all $i \geq 0$, once $\Gamma_J(M)$ is an $J$-torsion module. By using the fact that $\sqrt{\operatorname{ann}\left(\frac{\Gamma_J(M)}{\mathfrak{m}_0\Gamma_J(M)}\right)} = \sqrt{\mathfrak{m}_0 R + \operatorname{ann}(\Gamma_J(M))}$ and $\sqrt{\operatorname{ann}\left(\frac{M/JM}{\mathfrak{m}_0(M/JM)}\right)} = \sqrt{\mathfrak{m}_0 R + J + \operatorname{ann}(M)}$, one can conclude $\dim \frac{\Gamma_J(M)}{\mathfrak{m}_0\Gamma_J(M)} \leq \dim \frac{M/JM}{\mathfrak{m}_0(M/JM)} = d$. We use then [BH, Lemma 3.4] to obtain $H^i_{R_+,J}(\Gamma_J(M)) = 0$ for all $i > d$. Moreover, the exact sequence $0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0$ yields the long exact sequence

$$\cdots \to H^i_{R_+,J}(\Gamma_J(M)) \to H^i_{R_+,J}(M) \to H^i_{R_+,J}(M/\Gamma_J(M)) \to \cdots \tag{3.1}$$

We then derive

$$H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_J(M)),$$

for all $i > d$. We have

$$\dim \frac{M/\Gamma_J(M)}{(\mathfrak{m}_0 R + J)M/\Gamma_J(M)} = \dim \frac{M}{(\mathfrak{m}_0 R + J)M + \Gamma_J(M)} \leq d.$$

In this way, we can assume $M$ is an $J$-torsion-free module. So there exists an homogeneous element $a \in J$ which is a non zero-divisor on $M$. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ yields the long exact sequence

$$\cdots \to H^i_{R_+,J}(M) \xrightarrow{a} H^i_{R_+,J}(M) \to H^i_{R_+,J}(M/aM) \to \cdots$$

Once $\dim_R M/aM = n - 1$ and $\dim_R \frac{M/aM}{(\mathfrak{m}_0 R + J)M/aM} = d$, we use the inductive hypothesis to conclude $H^i_{R_+,J}(M/aM) = 0$ for all $i > d$. Then the above long exact sequence says $aH^i_{R_+,J}(M) = H^i_{R_+,J}(M)$. By Nakayama's Lemma, we obtain the desired result. $\square$

Next result is a similar result to [BH, Lemma 3.4(b)].

**Theorem 3.5.** *Let $J$ be a graded ideal of $R$. Then*

$$\dim M/(\mathfrak{m}_0 R + J)M = \sup\{i \;: H^i_{R_+,J}(M) \neq 0\}.$$

*Proof.* Consider the exact sequence

$$0 \to (\mathfrak{m}_0 R + J)M \to M \to M/(\mathfrak{m}_0 R + J)M \to 0,$$

which yields the long exact sequence

$$\cdots \to H^i_{R_+,J}(M) \to H^i_{R_+,J}(M/(\mathfrak{m}_0 R + J)M) \to H^{i+1}_{R_+,J}((\mathfrak{m}_0 R + J)M) \to \cdots.$$

Now note that

$$\dim \frac{(\mathfrak{m}_0 R + J)M}{(\mathfrak{m}_0 R + J)(\mathfrak{m}_0 R + J)M} \leq \dim \frac{M}{(\mathfrak{m}_0 R + J)^2 M} = \dim \frac{M}{(\mathfrak{m}_0 R + J)M} = d;$$

thus, because of Theorem 3.4, we obtain $H^i_{R_+,\mathfrak{b}}((\mathfrak{m}_0 R + J)M) = 0$ for $i > d$. On the other hand, by using [TYY, Corollary 2.5] and [HIO, Corollary 35.20], one sees

$$H^d_{R_+,J}(M/(\mathfrak{m}_0 R + J)M) \cong H^d_{R_+}(M/(\mathfrak{m}_0 R + J)M) \cong H^d_{(R/\mathfrak{m}_0 R)_+}(M/(\mathfrak{m}_0 R + J)M).$$

This last cohomology module is nonzero by [HIO, Corollary 36.19]. By observing the above long exact sequence we get the desired result. $\square$

# 4 Depth $(I, J)$ on graded module

In this section, we work with the concept of depth of a pair homogeneous ideals of $R$, and obtain an expression for it as $M$ and $R$ are both Cohen-Macaulay. For the ordinary case the [BS, Theorem 6.2.7] says

$$\operatorname{depth}_I(M) = \inf\{i \in \mathbb{N}_0 \ : \ H^i_I(M) \neq 0\},$$

for an $R$-ideal $I$ and a finite $R$-module $M$ such that $IM \neq M$. For the case of a pair $(I, J)$ ideals it was defined in [AAS, Definition 3.1]; we then recall this definition as follows.

Throughout this section we assume $R$ and $M$ are as in section 3.

**Definition 4.1.** *Let $I$, $J$ be two homogeneous ideals of the graded ring $R$ and $M$ a graded $R$-module. We define depth of $(I, J)$ on $M$ by*

$$\operatorname{depth}(I, J, M) = \inf\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \tilde{W}(I, J)\}$$

*it this infimum exists, and $\infty$ otherwise.*

**Remark 4.2.** In the general case, by [TYY, Theorem 4.1 and Theorem 3.2] or [AAS, Proposition 3.3], we have

$$\operatorname{depth}(I,J,M) = \inf\{i \in \mathbb{N}_0 | H^i_{I,J}(M) \neq 0\}.$$

Let $I, J, K$ be three homogeneous ideals of $R$. Let us introduce the number

$$\operatorname{ht}(I,J,K) := \inf\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in W(I,J) \cap V(K)\}.$$

Note that when $J = (0)$,

$$\operatorname{ht}(I,J,K) = \operatorname{ht}(I+K).$$

**Proposition 4.3.** *Assume $M$ is a Cohen-Macaulay $R$-module and $R$ is a Cohen-Macaulay ring. Set $I = \sqrt{\operatorname{ann}_R M}$. Then*

$$\operatorname{depth}(R_+, J, M) = \operatorname{ht}(R_+, I, J) - \operatorname{ht}(I).$$

*In particular,* $\operatorname{ht}(R_+, I, J) - \operatorname{ht}(I) = \inf\{i \in \mathbb{N}_0 \mid H^i_{R_+,J}(M) \neq 0\}$.

*Proof.* By [TYY, Theorem 4.1] (or [AAS, Proposition 3.3]),

$$\operatorname{depth}(R_+, J, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(R_+, J)\}.$$

Note that

$$\operatorname{depth}(R_+, J, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid \mathfrak{p} \in W(R_+, J) \cap V(I)\},$$

once $M_{\mathfrak{p}} = 0$ as $\mathfrak{p} \notin V(I)$, that is, $\operatorname{depth} M_{\mathfrak{p}} = \infty$. Since $M$ is Cohen-Macaulay, in particular,

$$\operatorname{depth} M_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$$

for each $\mathfrak{p} \in W(R_+, J)$. But $R$ is also Cohen-Macaulay by hypothesis, so $\dim M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}/I_{\mathfrak{p}} = \dim R_{\mathfrak{p}} - \operatorname{ht} I_{\mathfrak{p}}$. By [L, Lemma 1.2.2], all minimal primes of $I$ have the same height, so that, $\operatorname{ht} I_{\mathfrak{p}} = \operatorname{ht} I$ for each $\mathfrak{p}$. By combining the above arguments, we obtain

$$\operatorname{depth}(R_+, J, M) = \inf\{\dim R_{\mathfrak{p}} - \operatorname{ht} I \mid \mathfrak{p} \in W(R_+, J) \cap V(I)\}.$$

This last one equals $\operatorname{ht}(R_+, J, K) - \operatorname{ht} I$, by definition. The proof is completed. $\square$

**Proposition 4.4.** *Let $J$ be a graded ideal of $R$. Then any integer $i$ for which $H^i_{R_+,J}(M) \neq 0$ satisfies*

$$\operatorname{depth}(R_+, J, M) \leq i \leq \dim M/(\mathfrak{m}_0 R + J)M.$$

*Proof.* The proof follows by Remark 4.2 and Theorem 3.5. □

**Corollary 4.5.** *There is exactly one integer $i$ for which $H^i_{R_+,J}(M) \neq 0$ if and only if*

$$\operatorname{depth}(R_+, J, M) = \dim M/(\mathfrak{m}_0 R + J)M.$$

# 5 Basic finiteness, vanishing theorem and asymptotical stability

In this section, we generate two classical results about the graded components of local cohomology for the case of local cohomology with respect to a pair of ideals (Proposition 5.2 and Theorem 5.4). Finally, we obtain a result on the asymptotical stability of the sequence $\{\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$.

Throughout this section, we will make the following assumptions.

Let $R = \oplus_{d\geq 0} R_d$ denote a positively graded commutative Noetherian ring, which is standard, that is, $R = R_0[R_1]$ and let $M$ be a finitely generated module over $R$. Assume $R_0$ is a local ring of maximal ideal $\mathfrak{m}_0$ and residual field $R_0/\mathfrak{m}_0$ is infinite. Set $R_+ = \oplus_{i>0} R_i$, the irrelevant ideal of $R$.

It is well-known that $H^i_{R_+}(M)_n$ is finitely generated $R_0$-module for all $n \in \mathbb{Z}$ and $H^i_{R_+}(M)_n = 0$, for $n$ sufficiently large. Next assertion gives a positive answer for the case of cohomology modules with respect to a pair of ideals.

**Remark 5.1.** *Let $I, K, J$ be arbitrary ideals of $R$, and let $M$ be a $K$-torsion $R$-module. It is easy to check that $H^i_{I+K,J}(M) \cong H^i_{I,J}(M)$ for all $i \in \mathbb{N}_0$.*

**Proposition 5.2.** *Let $J$ be an ideal generated by elements of zero degree. Set $\mathfrak{b} = \mathfrak{b}_0 + R_+$. Then for $i \geq 1$,*

$$H^i_{\mathfrak{b},J}(M)_n = 0,$$

*for $n$ sufficiently large.*

*Proof.* Firstly we will prove the assertion as $\mathfrak{b}_0 = 0$. We proceed by induction on $\dim M$. If $\dim M = 0$ then by [TYY, Theorem 4.7(1)], the result is clearly true in this case. Suppose now $\dim M > 0$ and the result is established for finitely generated modules of dimension less than $\dim M$.

By [TYY, Corollary 2.5], $H^i_{R_+,J}(\Gamma_J(M)) \cong H^i_{R_+}(\Gamma_J(M))$, for all $i \geq 0$. The exact sequence $0 \to \Gamma_J(M) \to M \to M/\Gamma_J(M) \to 0$ yields then the long exact sequence

$$H^i_{R_+}(\Gamma_J(M))_n \to H^i_{R_+,J}(M)_n \to H^i_{R_+,J}(M/\Gamma_J(M))_n \to H^{i+1}_{R_+}(\Gamma_J(M))_n.$$

One can conclude that $H^i_{R_+,J}(M)_n$ is isomorphic to $H^i_{R_+,J}(M/\Gamma_J(M))_n$ for all $n$ sufficiently large. Hence, we can assume that $M$ is an $J$-torsion-free $R$-module. As $J$ is generated by homogeneous elements of zero degree, by a Prime Avoidance Lemma there exists an element $x$ in $J$ of zero degree which is a non zero divisor on $M$.

The exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

induces an exact sequence

$$H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_n \to H^i_{R_+,J}(M/xM)_n.$$

By induction, for every $i \geq 1$, $H^i_{R_+,J}(M/xM)_n = 0$ for all $n$ sufficiently large; the above exact sequence yields then an epimorphism

$$H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_n,$$

for all $n >> 0$, so

$$H^i_{R_+,J}(M)_n = xH^i_{R_+,J}(M)_n.$$

Nakayama's Lemma completes the first part.

Now we conclude the proposition. We proceed by induction on $\dim M$. If $\dim M = 0$, the result is clearly true.

Consider the exact sequence $0 \to \Gamma_{\mathfrak{b}_0}(M) \to M \to M/\Gamma_{\mathfrak{b}_0}(M) \to 0$, which yields, by Remark 5.1, the long exact sequence

$$H^i_{R_+}(\Gamma_{\mathfrak{b}_0}(M))_n \to H^i_{R_+,J}(M)_n \to H^i_{R_+,J}(M/\Gamma_{\mathfrak{b}_0}(M))_n \to H^{i+1}_{R_+}(\Gamma_{\mathfrak{b}_0}(M))_n.$$

So, by using the first part, we obtain the isomorphism

$$H^i_{R_+,J}(M)_n \to H^i_{R_+,J}(M/\Gamma_{\mathfrak{b}_0}(M))_n$$

for $n$ large. We then may assume there exists an element $x \in \mathfrak{b}_0$ which is a nonzero divisor on $M$. The final result is obtained by following the same arguments as in the last paragraph of the first part.

$\square$

**Lemma 5.3.** *If $J$ is generated by elements of zero degree and $\Gamma_{R_+,J}(M)_n = 0$ for $n >> 0$, then, for each $i \geq 0$, $H^i_{R_+,J}(M)_n$ is a finitely generated $R_0$-module for all integer $n$.*

*Proof.* We proceed by induction on $i$. By hypothesis and by the fact that $M_n$ for $n << 0$, we conclude that there exists $u \in \mathbb{N}$ such that $R_+^u H^0_{R_+,J}(M) = 0$. Note that $R_+^i H^0_{R_+,J}(M)/R_+^{i+1} H^0_{R_+,J}(M)$ is a Noetherian $R/R_+$-module, that is, a Noetherian $R_0$-module for each $i = 0, ..., u-1$. One then obtains $H^0_{R_+,J}(M)$ is a finitely generated $R_0$-module, and so is $H^0_{R_+,J}(M)_n$ for all $n \in \mathbb{Z}$.

Suppose now $i > 0$ and the result is established for smaller values of $i$. Because of the graded isomorphism $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$ for $i > 0$ (see [TYY, Corollary 1.13]), we can assume $M$ is an $(R_+, J)$-torsion-free $R$-module (and so is an $R_+$-torsion-free module). Then there exists an element $x \in R_+$ which is a non-zero divisor on $M$, say $\deg(x) = a$. The exact sequence

$$0 \to M \xrightarrow{x} M[a] \to (M/xM)[a] \to 0$$

induces an exact sequence

$$H^{i-1}_{R_+,J}(M)_{n+a} \to H^{i-1}_{R_+,J}(M/xM)_{n+a} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+a}.$$

From the above exact sequence and Proposition 5.2 one can deduce $\Gamma_{R_+,J}(M/xM)_n = 0$ for $n$ sufficiently large. From the inductive hypothesis it then follows that $H^{i-1}_{R_+,J}(M/xM)_q$ is finitely generated for all $q \in \mathbb{Z}$. Again by Proposition 5.2, there exists $s \in \mathbb{Z}$ such that $H^{i-1}_{R_+,J}(M/xM)_n = 0$ for all $n \geq s$ and $H^i_{R_+,J}(M)_n = 0$ for all $n \geq s - a$.

Fix $n \in \mathbb{Z}$ and let $k \geq 0$ be an integer such that $n + ka \geq s - a$. So $H^i_{R_+,J}(M)_{n+ka} = 0$. For each $j = 0, .., k-1$, we have the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+(j+1)a} \to H^i_{R_+,J}(M)_{n+ja} \xrightarrow{x} H^i_{R_+,J}(M)_{n+(j+1)a}.$$

In conclusion, we obtain $H^i_{R_+,J}(M)_{n+ja}$ is a finitely generated for $j = k-1, k-2, ..., 1, 0$, so that $H^i_{R_+,J}(M)_n$ is a finitely generated for all $n \in \mathbb{Z}$.

$\square$

**Theorem 5.4.** *If $J$ is generated by elements of zero degree, then, for all $i \geq 1$, $H^i_{R_+,J}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$.*

*Proof.* It follows from the graded isomorphism $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$ for $i > 0$ and the above lemma. $\square$

Consider the following definition. We say $\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is *asymptotically increasing for $n \to -\infty$*, if there exists an $n_0 \in \mathbb{Z}$ such that

$$\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1})$$

for all $n \leq n_0$.

**Lemma 5.5.** *Let $M$ be a finitely generated graded $R$-module and $J$ an arbitrary ideal. Let $i \in \mathbb{N}_0$ be such that $H^j_{R_+,J}(M)_n$ is finitely generated $R_0$-module for all $j < i$ and $n << 0$. Then $\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is asymptotically increasing for $n \to -\infty$.*

*Proof.* We prove by induction on $i$. The case $i = 0$ is trivial as $H^0_{R_+,J}(M)_n = 0$ for all $n \ll 0$. So, let $i > 0$. In view of the natural graded isomorphism $H^k_{R_+,J}(M) \cong H^k_{R_+,J}(M/\Gamma_{R_+,J}(M))$ for all $k \geq 1$ (see [TYY, Corollary 1.13 (4)]), we may assume $M$ is an $(R_+, J)$-torsion-free $R$-module. Since $\Gamma_{R_+}(M) \subseteq \Gamma_{R_+,J}(M)$, we now use [BS, Lemma 2.1.1] and [BH, Proposition 1.5.12] to deduce that $R_1$ contains an element $x$ which is a non zero-divisor on $M$. The exact sequence $0 \to M(-1) \xrightarrow{x} M \to M/xM \to 0$ induces a graded long exact sequence

$$H^{k-1}_{R_+,J}(M) \to H^{k-1}_{R_+,J}(M/xM) \to H^k_{R_+,J}(M)(-1) \to H^k_{R_+,J}(M).$$

Through this sequence, we obtain $H^{j-1}_{R_+,J}(M/xM)$ is finitely generated for all $j < i$. So, by induction, there exists some $n_1 \in \mathbb{Z}$ such that

$$\mathrm{Ass}_{R_0}(H^{j-1}_{R_+,J}(M/xM)_n) = \mathrm{Ass}_{R_0}(H^{j-1}_{R_+,J}(M/xM)_{n_1}) =: \mathcal{A} \text{ for all } n \leq n_1.$$

Moreover, there exists some $n_2 < n_1$ such that $H^{i-1}_{R_+,J}(M)_{n+1} = 0$ for all $n \leq n_2$. So, for each $n \leq n_2$, we have an exact sequence of $R_0$-modules

$$0 \to H^{i-1}_{R_+,J}(M/xM)_{n+1} \to H^i_{R_+,J}(M)_n \to H^i_{R_+,J}(M)_{n+1},$$

induced by the above exact sequence. This shows that

$$\mathcal{A} \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \mathcal{A} \cup \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1}), \text{ for all } n \leq n_2.$$

One can then conclude that

$$\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1}), \text{ for all } n < n_2.$$

□

**Theorem 5.6.** *Let $M$ be a finitely generated graded $R$-module, $J$ an ideal of $R$ generated by elements of zero degree, and let $i \in \mathbb{N}$ be such that $H^j_{R_+,J}(M)_n$ is finitely generated $R_0$-module for all $j < i$ and $n << 0$. Then there exists a finite subset $X$ of $\mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n) = X$ for $n << 0$.*

*Proof.* This is a consequence of Lemma 5.5 and Theorem 5.4, once $\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is finite for all integer $n$. □

If the sequence $\{\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$ satisfies the assertion in Theorem 5.6, we say it is *asymptotically stable for $n \to -\infty$*.

As an immediate consequence of Corollary 4.5, Lemma 5.5 and Theorem 5.6 we get the following result.

**Corollary 5.7.** *Let $M$ be a finitely generated graded $R$-module.*

(i) *If $\mathrm{depth}(R_+, J, M) = \dim M/(\mathfrak{m}_0 R + J)M$ then, for all $i \geq 0$, $\{\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$;*

(ii) *If $t = \mathrm{depth}(R_+, J, M)$, then $\{\mathrm{Ass}_{R_0}(H^t_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$.*

Last Corollary, item (ii), was showed in [BH, Proposition 5.6] for the case $J = 0$.

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